MATH 147, SPRING 2021: CONTENT AND PRACTICE PROBLEMS FOR EXAM 3

You should know the following topics for Exam 3.

(i) Triple integrals, including spherical coordinates, cylindrical coordinates, and the change of variables formula.
(ii) Arc length and line integrals of scalar and vector valued functions.
(iii) Surface area and surface integrals of scalar functions and vector fields.
(iv) The divergence of a vector field and the Divergence Theorem.

Practice Problems.

1. Calculate $\int\int\int_B y^2z^2\, dV$ for B the solid bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

2. Calculate $\int\int\int_B z^3\sqrt{x^2+y^2+z^2}\, dV$, for B the solid hemisphere with radius 1 and $z \geq 0$.

3. Let B_0 be the parallelepiped spanned by the vectors $2\vec{i} - \vec{j} + k, 3\vec{i} + k, 4\vec{j} - \vec{k}$ and B the parallelepiped obtained by translating the corner of B_0 at the origin to the point (3,2,1). Calculate $\int\int\int_B 2x - y + 3z\, dV$.

4. Let C be a curve that starts at the north pole of the sphere of radius R centered at the origin and ends at the south pole. You can choose any such C. Find a parametrization of C and calculate the line integrals $\int_C x + y + z\, ds$ and $\int_C \mathbf{F} \cdot d\mathbf{r}$, for $\mathbf{F} = xi + yj + zk$.

5. Let S denote the top half of the sphere of radius R centered at the origin and consider the scalar function $f(x, y, z) = \sqrt{1 + x^2 + y^2 + z^2}$. Calculate $\int\int_S f(x, y, z)\, dS$ in two ways :

(i) Intuitively, by just writing down your answer. You must justify this.
(ii) Directly, using a parametrization of S.

6. For the vector field $\vec{F} = 3z^2\vec{k}$, use the limit definition of the divergence with decreasing spheres to show that $Div(\vec{F})(P) = 6z_0$, for any point $P = (x_0, y_0, z_0)$.

7. Let S be that portion of the plane $x + y - z = 0$, with $0 \leq x \leq a$ and $0 \leq y \leq b$, for $a, b > 0$ and let $\vec{F} = x\vec{i} + 2y\vec{j} + 3\vec{k}$. Show by direct calculation that the surface integral $\int\int_S \vec{F} \cdot d\vec{S}$ can change, depending upon the parametrization of S. Be sure to exhibit explicitly two different parametrizations of S.

8. Let S be that portion of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$.

(i) Calculate the surface area of S.
(ii) Calculate the surface integral $\int\int_S x^2 + y^2\, dS$.

9. Let B be the solid bounded by the closed surface S which is that portion of the cone $z = \sqrt{x^2+y^2}$, with $1 \leq z \leq 2$. verify the Divergence Theorem for $\mathbf{F} = xi + yj + z^2k$.

10. Let B denote the solid sphere of radius R centered at the origin, and let $P = (0, 0, R)$ denote the north pole. Find the average value of the distance of points $(x, y, z) \in B$ to the point P.

11. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = x^2yi + y^2zj + z^2xk$, for $C : \mathbf{r}(t) = e^{-t}i + e^{-2t}j + e^{-3t}k$, with $0 \leq t < \infty$.

12. Find the volume of the solid W, whose boundary is a closed surface, if

$$\int\int_{\partial W} \{(x + xy + z)i + (x + 3y - \frac{1}{2}y^2)j + 4zk\} \cdot d\mathbf{S} = 16.$$